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FOUR GENERAL CONTINUITY PROPERTIES, FOR PAIRS OF FUNCTIONS, RELATIONS AND RELATORS, WHOSE PARTICULAR CASES COULD BE INVESTIGATED BY HUNDREDS OF MATHEMATICIANS

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Abstract. This is a research proposal for those who are interested in the unification of several continuity-like properties of functions and relations in the framework of relator spaces. For this, motivated by Galois connections, we shall use a pair of relations instead of a single function or relation.

A family $\mathcal{R}$ of relations on one set X to another Y is called a relator on X to Y. All reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be derived from relators. Therefore, they should not be studied separately.

From the various topological and algebraic stuctures (such as lower bounds, minimum and infimum, for instance) derived from relators, by using Pataki connections, we obtain several closure and modification operations for relators. Each of them leads to four reasonable continuity or increasingness properties.

1. Relations and functions

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation F on X to itself is simply called a relation on X. And, $\Delta_X = \{ (x, x) : x \in X \}$ is called the *identity relation of* X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* or *neighbourhoods* of x and A under F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x F y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_F = \{ x \in X : F(x) \neq \emptyset \}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *total relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function of* F if $f(x) \in F(x)$ for all $x \in D_R$. Thus, by the Axiom of Choice [19], we can see that every relation is the union of its selection functions.

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For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. In [76, 81], they were briefly called corelations on X to Y.

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation on X to Y* [86, 41]. Thus, closures (proximities) [93] are super (hyper) relations.

Note that a super relation on X to Y is an arbitrary subset of $\mathcal{P}(X) \times Y$. While, a corelation on X to Y is a particular subset of $\mathcal{P}(X) \times \mathcal{P}(Y)$. Thus, set inclusion is a natural partial order for super relations, but not for corelations.

For a relation F on X to Y, the relation, $F^c = (X \times Y) \setminus F$ is called the *complement* of F. Thus, it can be shown that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$, and $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$.

Moreover, the relation $F^{-1} = \{(y, x) : \ (x, y) \in F\}$ is called the *inverse* of F. Thus, it can be shown that $F^{-1}[B] = \{x \in X : \ F(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$, and in particular $D_F = F^{-1}[Y]$.

If F is a relation on X to Y, then we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, if G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, it can be shown that $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$ [74].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

2. Important relational properties

Now, a relation R on X, i.e., a subset R of X^2, may be briefly defined to be *reflexive* and *transitive* if under the plausible notations $R^0 = \Delta_X$ and $R^2 = R \circ R$ we have $R^0 \subseteq R$ and $R^2 \subseteq R$, respectively.

Moreover, R may be briefly defined to be *symmetric* and *antisymmetric* if $R^{-1} \subseteq R$ and $R \cap R^{-1} \subseteq R^0$, respectively. And, R may be briefly defined to be *total* and *directive* if $X^2 \subseteq R \cup R^{-1}$ and $X^2 \subseteq R^{-1} \circ R$, respectively.

In addition to the above properties, several further remarkable relational properties were studied in [58]. For instance, a relation R on X was called *quasi-antisymmetric* if $y \in R(x)$ and $x \in R(y)$ imply $R(x) = R(y)$ for all $x, y \in X$.

Much more importantly, a relation R on X was called *non-mingled-valued* if $R(x) \cap R(y) \neq \emptyset$ implies $R(x) = R(y)$ for all $x, y \in X$. Thus, it can be shown that all equivalence and linear relations [89] are non-mingled-valued.

The latter two properties, by using the reasonable notations $R^- = R^{-1} \circ R$ and $R^\circ = \big(R^{-1} \circ R^c\big)^c$, can be reformulated in the forms $R \cap R^{-1} \subseteq R^\circ$ and $R \circ R^- \subseteq R$, respectively. Note that if R is non-partial, then $R \subseteq R \circ R^-$ holds.

In the sequel, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation.* And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation.*

According to general algebra, for any relation R on X, we may naturally define $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$. Thus, R^{∞} is the smallest preorder relation containing R [17].

Now, in contrast to $(F^c)^c = F$ and $(F^{-1})^{-1} = F$, we have $(R^{\infty})^{\infty} = R^{\infty}$. Moreover, analogously to $(F^c)^{-1} = (F^{-1})^c$, we also have $(R^{\infty})^{-1} = (R^{-1})^{\infty}$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup (A^c \times X)$ is an important preorder on X [38]. While, for a *pseudo-metric* d on X, the *Weil surrounding* $B_r^d = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [95].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Now, as a straightforward generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = (A \times B) \cup (A^c \times Y)$ [18]. (See also [10, pp. 42 and 351].)

However, it is more interesting to note that if $\mathcal{A} = (A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}(X)$, then the *Cantor relation* $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^{\infty} \big(A_n \times A_n^c\big)$ is also an important preorder on X [34, 20].

Note that if R is only reflexive relation on X and $x \in X$, then $\mathcal{A}_R(x) = \big(R^n(x)\big)_{n=1}^{\infty}$ is already an increasing sequence in $\mathcal{P}(X)$. Thus, the preorder relation $R_{\mathcal{A}_R(x)}$ may also be naturally investigated.

Moreover, for a real function φ of X and a quasi-pseudo-metric d on X [15], the *Brøndsted relation* $R_{(\varphi, d)} = \{(x, y) \in X^2 : \ d(x, y) \leq \varphi(y) - \varphi(x)\}$ is also an important preorder on X [7] .

From this relation, by letting φ and d to be the zero functions, we can obtain the *specialization* and *preference relations* $R_d = \{(x, y) \in X^2 : \ d(x, y) = 0\}$ and $R_{\varphi} = \{(x, y) \in X^2 : \ \varphi(x) \leq \varphi(y)\}$, respectively. (See [9] and [94].)

3. Relator spaces

A family $\mathcal{R}$ of relations on one set X to another Y will be called a *relator on X to Y*, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ will be called a *relator space*. For the origins of this notion, see [48, 61], and the references in [48].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on X*. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [11, 43] and *uniform spaces* [93, 15]. However, they are insufficient for some important purposes. (See, [16] and [61, 74, 76, 91].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [16] and [78], respectively.

Moreover, a relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance and equivalence relators.*

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{R_A : \ A \in \mathcal{A}\}$, where $R_A = A^2 \cup (A^c \times X)$, is an important preorder relator on X. Such relators were first explicitly used by Pervin [38] and Levine [28].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : \ r > 0, \ d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : \ d(x, y) < r\}$, is an important tolerance relator on X. Such relators were already considered by Weil [95].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{S_{\mathcal{A}} : \ \mathcal{A} \in \mathfrak{S}\}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an important tolerance (equivalence) relator on X. Equivalence relators were studied by Levine [27].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we may naturally define $\mathcal{R}^{\star} = \big\{ R^{\star} : \ R \in \mathcal{R} \big\}$. However, this plausible notation may cause confusions if $\star$ is a set-theoretic operation.

For instance, for any relator $\mathcal{R}$ on X to Y, we may naturally define the *elementwise complement* $\mathcal{R}^c = \{R^c : \ R \in \mathcal{R}\}$, which may easily be confused with the *global complement* $\mathcal{R}^c = \mathcal{P}(X \times Y) \setminus \mathcal{R}$ of $\mathcal{R}$.

However, for instance, the practical notations $\mathcal{R}^{-1} = \{R^{-1} : \ R \in \mathcal{R}\}$, and $\mathcal{R}^{\infty} = \{R^{\infty} : \ R \in \mathcal{R}\}$ whenever $\mathcal{R}$ is only a relator on X, will certainly not cause any confusion in the sequel.

For a relator $\mathcal{R}$ on X, we may also define $\mathcal{R}^{\partial} = \big\{ S \subseteq X^2 : \ \ S^{\infty} \in \mathcal{R} \big\}$. Namely, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have $\mathcal{R}^{\infty} \subseteq \mathcal{S} \iff \mathcal{R} \subseteq \mathcal{S}^{\partial}$. That is, ∞ and ∂ form a Galois connection [11, p. 155].

The operations ∞ and ∂ were introduced by Mala [29, 31] and Pataki [36, 37], respectively. These two former PhD students of mine together with János Kurdics [23, 25], have made substantial developments in the theory of relators.

Moreover, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \big\{ R * S : \ \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}$. However, this notation may again cause confusions if $*$ is a set-theoretic operation

Therefore, in the former papers, we rather wrote $\mathcal{R} \wedge \mathcal{S} = \big\{ R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}$. Moreover, for instance, we also wrote $\mathcal{R} \triangle \mathcal{R}^{-1} = \big\{ R \cap R^{-1} : \ \ R \in \mathcal{R} \big\}$. Thus, $\mathcal{R} \triangle \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangle \mathcal{R}^{-1} \subseteq \mathcal{R} \wedge \mathcal{R}^{-1}$.

A function $\Box$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\Box} = \Box(\mathcal{R})$ is a relator on X to Y (on Y to X).

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

Concerning structures and unary operations for relators, we can freely use some basic terminology on set-to-set functions [81]. However, for closures and projections, we can now also use the terms *refinements and modifications*, respectively.

For instance, c and -1 are *involution operations* for relators. While, ∞ and ∂ are *projection operations* for relators. Moreover, the operation $\Box = c$, ∞ or ∂ is *inversion compatible* in the sense that $\big(\mathcal{R}^{\Box}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\Box}$.

While, if for instance $\operatorname{int}_{\mathcal{R}}(B) = \{x \in X : \ \exists\, R \in \mathcal{R} : \ R(x) \subseteq B\}$ for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathcal{R}) = \operatorname{int}_{\mathcal{R}}$, is a union-preserving structure for relators.

The first basic problem in the theory of relators is that, for any union-preserving structure $\mathfrak{F}$, we have to find an unary operation $\Box$ for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y we could have $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathcal{S} \subseteq \mathcal{R}^{\Box}$.

By using *Pataki connections* [36, 84], several closure operations can be derived from union-preserving structures. However, more generally, one can find first the *Galois adjoint* $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\Box_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$ [66].

By finding the Galois adjoint of the structure $\mathfrak{F}$, the second basic problem for relators, that which structures can be derived from relators, can also be solved. However, for this, some direct methods can also be well used [53, 68].

Now, for an operation $\Box$ for relators, a relator $\mathcal{R}$ on X to Y may be naturally called $\Box$*–fine* if $\mathcal{R}^{\Box} = \mathcal{R}$. And, for some structure $\mathfrak{F}$ for relators, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, for a structure $\mathfrak{F}$ for relators, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_R$ for some relation R on X to Y. Thus, singleton relators have to be actually called *properly simple.*

4. Topological structures derived from relators

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$ we define:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$; (4) $x \in \sigma_{\mathcal{R}}(y)$ if $x \in \operatorname{int}_{\mathcal{R}}(\{y\})$;

(5) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$; (6) $x \in \rho_{\mathcal{R}}(y)$ if $x \in \operatorname{cl}_{\mathcal{R}}(\{y\})$;

(7) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (8) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we also define:

(9) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$, (10) $A \in \tau\!\!\!-_{R}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$,

(11) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$, (12) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;

(13) $A \in \mathcal{N}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$; (14) $A \in \mathcal{M}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{D}_{\mathcal{R}}$.

The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called the *proximal, topological* and *infinitesimal interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ are called the *proximally open, topologically open, fat* and *rare (nowhere dense) subsets* of the relator space $X(\mathcal{R})$, respectively.

The origins of the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [13] and Smirnov's strong inclusion $\Subset$ [46], respectively. The families $\tau_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ were first explicitly used by the present author [53]. In particular, the practical notation $\tau\!\!\!-_{\mathcal{R}}$ has been suggested by János Kurdics.

By the above definitions, for any relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, we have

$$\operatorname{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}\big(B^c\big); \qquad \operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big);$$

$$\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}; \qquad \operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X;$$

$$\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \ D^c \notin \mathcal{E}_{\mathcal{R}} \big\} = \big\{ D \subseteq Y : \ \forall\ E \in \mathcal{E}_{\mathcal{R}} : \ E \cap D \neq \emptyset \big\};$$

where $\mathcal{C}_X(A) = X \setminus A$ for all $A \subseteq X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then we also have

$$\tau\!\!\!-_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}; \qquad \tau\!\!\!-_{\mathcal{R}} = \big\{ A \subseteq X : \ A^c \in \tau_{\mathcal{R}} \big\};$$

$$\mathcal{F}_{\mathcal{R}} = \big\{ A \subseteq X : \ A^c \in \mathcal{T}_{\mathcal{R}} \big\}; \qquad \mathcal{M}_{\mathcal{R}} = \big\{ A \subseteq X : \ A^c \in \mathcal{N}_{\mathcal{R}} \big\}.$$

Thus, the proximal closures and proximally open sets are usually more convenient tools, than the topological closures (proximal interiors) and topologically open sets, respectively.

Moreover, the fat sets are frequently also more convenient tools than the topologically open ones. For instance, if $\leq$ is a relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are just the families of all *ascending and residual subsets* of the goset $X(\leq)$, respectively.

Moreover, if in particular $X = \mathbb{R}$ and R is a relation on X such that

$$R(x) = \{ x - 1 \} \cup [\, x\,, +\infty \,[$$

for all $x \in X$, then $\mathcal{T}_R = \{ \emptyset, X \}$, but $\mathcal{E}_R$ is quite large family.

However, the importance of the fat and dense sets lies manly in the fact that if $\mathcal{R}$ is a relator on X to Y, and φ and ψ are functions of a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, then by using the function (φ, ψ), defined by

$$(\varphi, \psi)(\gamma) = \big(\varphi(\gamma), \psi(\gamma)\big)$$

for all $\gamma \in \Gamma$, we may also define

(15) $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$;

(16) $\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Now, for any $x \in X$, we may also naturally define:

(17) $x \in \operatorname{lim}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \operatorname{Lim}_{\mathcal{R}}(\psi)$; (18) $x \in \operatorname{adh}_{\mathcal{R}}(\Psi)$ if $x_{\Gamma} \in \operatorname{Adh}_{\mathcal{R}}(\psi)$;

where x_{Γ} is a function of Γ to X such that $x_{\Gamma}(\gamma) = x$ for all $\gamma \in \Gamma$.

The *big limit relation* $\operatorname{Lim}_{\mathcal{R}}$, suggested by Efremović and Švarc [14], is, in general, a much stronger tool in the relator space $(X, Y)(\mathcal{R})$ than the *big closure and interior relations* $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ suggested by Efremović [13] and Smirnov [46].

Namely, it can be shown that, for any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if and only if there exist a preordered set $\Gamma(\leq)$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ $\big(\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)\big)$.

5. Algebraic structures derived from relators

If $\mathcal{R}$ is a relator on X to Y, then according to [63], for any $A \subseteq X$, $B \subseteq Y$, $x \in X$ and $y \in Y$ we may also naturally define:

(1) $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ and $B \in \operatorname{Ub}_{\mathcal{R}}(A)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$;

(2) $x \in \operatorname{lb}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Lb}_{\mathcal{R}}(B)$; (3) $y \in \operatorname{ub}_{\mathcal{R}}(A)$ if $\{y\} \in \operatorname{Ub}_{\mathcal{R}}(A)$;

(4) $B \in \mathfrak{L}_{\mathcal{R}}$ if $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$; (5) $A \in \mathfrak{U}_{\mathcal{R}}$ if $\operatorname{ub}_{\mathcal{R}}(A) \neq \emptyset$.

Moreover, in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we may also naturally define:

(6) $\min_{\mathcal{R}}(A) = A \cap \operatorname{lb}_{\mathcal{R}}(A)$; (7) $\max_{\mathcal{R}}(A) = A \cap \operatorname{ub}_{\mathcal{R}}(A)$;

(8) $\operatorname{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Lb}_{\mathcal{R}}(A)$; (9) $\operatorname{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \operatorname{Ub}_{\mathcal{R}}(A)$;

(10) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\operatorname{lb}_{\mathcal{R}}(A)\big)$; (11) $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\operatorname{ub}_{\mathcal{R}}(A)\big)$;

(12) $\operatorname{Inf}_{\mathcal{R}}(A) = \operatorname{Max}_{\mathcal{R}}\big[\operatorname{Lb}_{\mathcal{R}}(A)\big]$; (13) $\operatorname{Sup}_{\mathcal{R}}(A) = \operatorname{Min}_{\mathcal{R}}\big[\operatorname{Ub}_{\mathcal{R}}(A)\big]$;

(14) $A \in \ell_{\mathcal{R}}$ if $A \in \operatorname{Lb}_{\mathcal{R}}(A)$; (15) $A \in u_{\mathcal{R}}$ if $A \in \operatorname{Ub}_{\mathcal{R}}(A)$;

(16) $A \in \mathcal{L}_{\mathcal{R}}$ if $A \subseteq \operatorname{lb}_{\mathcal{R}}(A)$; (17) $A \in \mathcal{U}_{\mathcal{R}}$ if $A \subseteq \operatorname{ub}_{\mathcal{R}}(A)$.

By the above definitions, for any relator $\mathcal{R}$ on X to Y, we have

$$\operatorname{Ub}_{\mathcal{R}} = \operatorname{Lb}_{\mathcal{R}^{-1}} = \operatorname{Lb}_{\mathcal{R}}^{-1}; \qquad \operatorname{ub}_{\mathcal{R}} = \operatorname{lb}_{\mathcal{R}^{-1}}; \qquad \mathfrak{U}_{\mathcal{R}} = \mathfrak{L}_{\mathcal{R}^{-1}}.$$

Moreover, if in particular $\mathcal{R}$ is a relator on X, then we also have

$$\ell_{\mathcal{R}} \subseteq \mathcal{L}_{\mathcal{R}} \cap \mathcal{U}_{\mathcal{R}}; \qquad u_{\mathcal{R}} = \ell_{\mathcal{R}^{-1}} = \ell_{\mathcal{R}}; \qquad \mathcal{U}_{\mathcal{R}} = \mathcal{L}_{\mathcal{R}^{-1}};$$

$$A \in \ell_{\mathcal{R}} \iff A \in \operatorname{Min}_{\mathcal{R}}(A) \iff A \in \operatorname{Inf}_{\mathcal{R}}(A);$$

$$\ell_{\mathcal{R}} = \operatorname{Min}_{\mathcal{R}}\big[\mathcal{P}(X)\big]; \qquad \mathcal{L}_{\mathcal{R}} = \min_{\mathcal{R}}\big[\mathcal{P}(X)\big].$$

However, the above algebraic structures are not independent of the former topological ones. Namely, if R is a relation on X to Y, then for any $A \subseteq X$ and

$B \subseteq Y$ we have

$$A \times B \subseteq R \iff \forall\ a \in A: \ B \subseteq R(a) \iff \forall\ a \in A: \ R(a)^{c} \subseteq B^{c}$$
$$\iff \forall\ a \in A: \ R^{c}(a) \subseteq B^{c} \iff R^{c}[A] \subseteq B^{c}.$$

Therefore, if $\mathcal{R}$ is a relator on X to Y, then by the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$, we also have

$$A \in \mathrm{Lb}_{\mathcal{R}}(B) \iff A \in \mathrm{Int}_{\mathcal{R}^c}(B^c) \iff A \in \big(\mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y\big)(B).$$

Hence, we can already infer that

$$\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y; \quad \mathrm{lb}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y; \quad \mathrm{Int}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y; \quad \mathrm{int}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y.$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other by the above equalities as the exponential and the trigonometric functions are by the celebrated Euler formulas [47, p. 227].

6. Closure and projection operations for relators

From the various structures derived from relators, by using Pataki connections [36, 84], we can derive several closure operations for relators.

However, the first three of the following operations were already considered by Bourbaki [6, p. 169], Kenyon [22] and H. Nakano and K. Nakano [32].

For instance, for any relator $\mathcal{R}$ on X to Y, the relators

$$\mathcal{R}^{*} = \big\{ S \subseteq X \times Y: \quad \exists\ R \in \mathcal{R}: \quad R \subseteq S \big\};$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y: \quad \forall\ A \subseteq X: \quad \exists\ R \in \mathcal{R}: \quad R[A] \subseteq S[A] \big\};$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y: \quad \forall\ x \in X: \quad \exists\ R \in \mathcal{R}: \quad R(x) \subseteq S(x) \big\};$$
$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y: \quad \forall\ x \in X: \quad \exists\ u \in X: \quad \exists\ R \in \mathcal{R}: \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological, and paratopological closures (refinements)* of the relator $\mathcal{R}$, respectively.

Thus, we evidently have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\triangle}$ for any relator $\mathcal{R}$ on X to Y. Moreover, if in particular $\mathcal{R}$ is a relator on X, then we can easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

However, it is now more important to note that, because of the corresponding definitions in Section 4, we also have

$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y: \quad \forall\ A \subseteq X: \quad A \in \mathrm{Int}_{R}\big(S[A]\big) \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y: \quad \forall\ x \in X: \quad x \in \mathrm{int}_{R}\big(S(x)\big) \big\},$$
$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y: \quad \forall\ x \in X: \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.$$

Morover, by using a Pataki connections [36, 84], we can, for instance, prove the following theorems and their corollaries in a unified way.

Theorem 6.1. $\#$, $\wedge$ *and* $\triangle$ *are closure operations for relators such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y, *we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#} \subseteq \mathcal{R}^{\#} \iff \mathrm{Int}_{\mathcal{S}} \subseteq \mathrm{Int}_{\mathcal{R}} \iff \mathrm{Cl}_{\mathcal{R}} \subseteq \mathrm{Cl}_{\mathcal{S}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge} \iff \mathrm{int}_{\mathcal{S}} \subseteq \mathrm{int}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}} \subseteq \mathrm{cl}_{\mathcal{S}}$,

(3) $\mathcal{S} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{S}^{\triangle} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Corollary 6.2. *For any relator* $\mathcal{R}$ *on* X *to* Y,

(1) $\mathcal{S}=\mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Int}_{\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}$ *, or equivalently* $\operatorname{Cl}_{\mathcal{S}}=\operatorname{Cl}_{\mathcal{R}}$ *;*

(2) $\mathcal{S}=\mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{int}_{\mathcal{S}}=\operatorname{int}_{\mathcal{R}}$ *, or equivalently* $\operatorname{cl}_{\mathcal{S}}=\operatorname{cl}_{\mathcal{R}}$ *;*

(3) $\mathcal{S}=\mathcal{R}^{\triangle}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}}=\mathcal{E}_{\mathcal{R}}$ *, or equivalently* $\mathcal{D}_{\mathcal{S}}=\mathcal{D}_{\mathcal{R}}$ *.*

Theorem 6.3. $\#\,\partial$ *is a closure operation for relators such that for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*, we have*

$$\mathcal{S}\subseteq\mathcal{R}^{\#\,\partial}\iff\mathcal{S}^{\#\,\partial}\subseteq\mathcal{R}^{\#\,\partial}\iff\tau_{\mathcal{S}}\subseteq\tau_{\mathcal{R}}\iff\bar{\tau}_{\mathcal{S}}\subseteq\bar{\tau}_{\mathcal{R}}\,.$$

Corollary 6.4. *For any relator* $\mathcal{R}$ *on* X*,* $\mathcal{S}=\mathcal{R}^{\#\,\partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{S}}=\tau_{\mathcal{R}}$ *or equivalently* $\bar{\tau}_{\mathcal{S}}=\bar{\tau}_{\mathcal{R}}$ *.*

Remark 6.5. $\wedge\,\partial$ is only a preclosure operation for relators. Moreover, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{S}$ such that $\mathcal{T}_{\mathcal{S}}=\mathcal{T}_{\mathcal{R}}$. (See Mala [29, Example 5.3] and Pataki [36, Example 7.2].)

In the light of this and other disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in topology and analysis are based on open sets suggested by Tietze [92] and standardized by Bourbaki [6] and Kelley [21].

Moreover, it also a striking fact that, despite the results of Pervin [38], Fletcher and Lindgren [15] and the present author [68], generalized topologies and minimal structures are still intensively investigated by a great number of mathematicians.

Concerning the structures $\mathcal{T}$ and $\mathcal{F}$, instead of an analogue of Theorem 6.3, we can only prove the following generalizations of the results of Mala [29, 31].

Theorem 6.6. $\wedge\infty$ *is a modification operation for relators such that, for any two nonvoid relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*, we have*

$$\mathcal{S}^{\wedge\infty}\subseteq\mathcal{R}^{\wedge}\iff\mathcal{S}^{\wedge\infty}\subseteq\mathcal{R}^{\wedge\infty}\iff\mathcal{T}_{\mathcal{S}}\subseteq\mathcal{T}_{\mathcal{R}}\iff\mathcal{F}_{\mathcal{S}}\subseteq\mathcal{F}_{\mathcal{R}}\,.$$

Corollary 6.7. *For any nonvoid relator* $\mathcal{R}$ *on* X*,* $\mathcal{S}=\mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{S}}=\mathcal{T}_{\mathcal{R}}$ *or equivalently* $\mathcal{F}_{\mathcal{S}}\subseteq\mathcal{F}_{\mathcal{R}}$ *.*

Remark 6.8. Quite similar theorems can be proved concerning the modification operations $\#\infty$ and $\infty\#$.

Their advantage over the closure operation $\#\,\partial$ lies mainly in the fact that, in contrast to the letter one, they are still *stable* in the sense that they leave the relator $\{X^2\}$ fixed for any set X.

Finally, we note that, by using the notations

$$\circledast=c\,\#\,c\qquad\text{and}\qquad\circledwedge=c\wedge c$$

we can also prove the following analogues of Theorem 6.1 and its corollary.

Theorem 6.9. $\circledast$ *and* $\circledwedge$ *are closure operations for relators such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have*

(1) $\mathcal{S}\subseteq\mathcal{R}^{\circledast}\iff\mathcal{S}^{\circledast}\subseteq\mathcal{R}^{\circledast}\iff\operatorname{Lb}_{\mathcal{S}}\subseteq\operatorname{Lb}_{\mathcal{R}}$,

(2) $\mathcal{S}\subseteq\mathcal{R}^{\circledwedge}\iff\mathcal{S}^{\circledwedge}\subseteq\mathcal{R}^{\circledwedge}\iff\operatorname{lb}_{\mathcal{S}}\subseteq\operatorname{lb}_{\mathcal{R}}$.

Corollary 6.10. *For any relator* $\mathcal{R}$ *on* X *to* Y*,*

(1) $\mathcal{S}=\mathcal{R}^{\circledast}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Lb}_{\mathcal{S}}=\operatorname{Lb}_{\mathcal{R}}$ *;*

(2) $\mathcal{S}=\mathcal{R}^{\circledwedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{lb}_{\mathcal{S}}=\operatorname{lb}_{\mathcal{R}}$ *.*

Remark 6.11. In contrast to the operation $*$ and $\#$, the operations $\wedge$ and $\triangle$ are are not inversions compatible [52, 30]. Therefore, we shall also need the notations

$$\vee=\wedge-1\qquad\text{and}\qquad\nabla=\triangle\,-1\,.$$

Moreover, these operations are only left composition compatible in the sense that, under the notation $\Box = \wedge$ or $\triangle$, we have

$$\left(\mathcal{S} \circ \mathcal{R} \right)^{\Box} = \left(\mathcal{S} \circ \mathcal{R}^{\Box} \right)^{\Box}$$

for any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

7. Relational characterizations of increasing functions

To motivate our forthcoming, unifying definition for continuity properties, we shall start with some simple observations on increasing functions.

Definition 7.1. Assume that

(a) $X(R)$ and $Y(S)$ are gosets; (b) f is a function of X to Y.

Then, the function f is called *increasing* if for any $u, v \in X$

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v)\,.$$

Remark 7.2. Increasing functions have several useful characterizations in terms of the relations ub and max.

For instance, we can easily see that f is increasing if and only $f\,[\,\mathrm{ub}_R(A)\,] \subseteq \mathrm{ub}_S\big(\,f\,[\,A\,]\big)$ for all $A \subseteq X$ [78].

However, it is now more important to note that the following two theorems are also true.

Theorem 7.3. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $(u, v) \in R \implies \big(f(u), f(v)\big) \in S$

(3) $v \in R(u) \implies f(v) \in S\big(f(u)\big)$ *for all* $u \in X$.

Theorem 7.4. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $f \circ R \subseteq S \circ f$, (3) $R \subseteq f^{-1} \circ S \circ f$;

(4) $f \circ R \circ f^{-1} \subseteq S$, (5) $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. By the corresponding definitions, it is clear that, for any $u \in X$, the following assertions are equivalent:

$$v \in R(u) \implies f(v) \in S\big(f(u)\big)\,;$$

$$f\,[R(u)\,] \subseteq S\big(f(u)\big)\,; \qquad (f \circ R)(u) \subseteq (S \circ f)(u)\,.$$

Therefore, by Theorem 7.3, assertions (1) and (2) are also equivalent.

The proofs of the remaining equivalences depend on the increasingness and associativity of composition, and the inclusions

$$\Delta_X \subseteq f^{-1} \circ f \qquad \text{and} \qquad f \circ f^{-1} \subseteq \Delta_Y\,,$$

where Δ_X and Δ_Y are the identity functions of X and Y, respectively.

Remark 7.5. The latter inclusions indicate that assertions (2)–(5) need not be equivalent for an arbitrary relation f on $X(R)$ to $Y(S)$.

Therefore, they can be naturally used to define different increasingness properties of a relation F on $X(R)$ to $Y(S)$.

Remark 7.6. Having in mind the associated set-valued function, a relation F on the goset $X(R)$ to a set Y may be naturally called increasing if $u\,R\,v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$.

Thus, it can be easily shown that the relation F is increasing if and only if its inverse F^{-1} is *ascending-valued* in the sense that $F^{-1}(y)$ is an ascending subset of $X(R)$ for all $y \in Y$.

By using the corresponding definitions, the latter statement can be reformulated in the form that $R\,[\,F^{-1}(y)\,] \subseteq F^{-1}(y)$ for all $y \in Y$. That is, $R \circ F^{-1} \subseteq F^{-1}$, and thus $R \circ F^{-1} \subseteq F^{-1} \circ \Delta_Y$.

Remark 7.7. In addition to the above inclusion-increasingness, a relation F on $X(R)$ to $Y(S)$ may be naturally called *order-increasing* if $u \in \mathrm{lb}_R(v)$ implies $F(u) \in \mathrm{Lb}_S\big(F(v)\big)$ for all $v \in X$.

That is, $(u, v) \in R$ implies $F(u) \times F(v) \subseteq S$. Thus, it can be shown that F is order-increasing if and only if $F \circ R \circ F^{-1} \subseteq S$, or equivalently $F\,[R(u)] \subseteq \mathrm{ub}_S\big(F(u)\big)$ for all $u \in X$.

Now, as an immediate consequence of Theorem 7.4 and a basic theorem on box products, we can also state

Corollary 7.8. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $(f \boxtimes f)\,[R] \subseteq S$;

(3) $\big(f \boxtimes R\big)\,[\Delta_X] \subseteq (S \boxtimes f)^{-1}\,[\Delta_Y]$;

(4) $R \subseteq (f \boxtimes f)^{-1}\,[S]$;

(5) $\big(R^{-1} \boxtimes f\big)\,[\Delta_X] \subseteq \big(f^{-1} \boxtimes S\big)\,[\Delta_Y]$.

However, it is now more important to note that, by using our former operations on relators, Theorem 7.4 can be reformulated in the following instructive form.

Theorem 7.9. *Under the notations*

$$\mathcal{F} = \{f\}, \qquad \mathcal{R} = \{R\}, \qquad \mathcal{S} = \{S\}$$

the following assertions are equivalent:

(1) f *is increasing;*

(2) $\big(\mathcal{S}^* \circ \mathcal{F}^*\big)^* \subseteq \big(\mathcal{F}^* \circ \mathcal{R}^*\big)^*$,

(3) $\Big(\big(\mathcal{F}^*\big)^{-1} \circ \mathcal{S}^* \circ \mathcal{F}^*\Big)^* \subseteq \mathcal{R}^{**}$,

(4) $\mathcal{S}^{**} \subseteq \Big(\mathcal{F}^* \circ \mathcal{R}^* \circ \big(\mathcal{F}^*\big)^{-1}\Big)^*$,

(5) $\Big(\big(\mathcal{F}^*\big)^{-1} \circ \mathcal{S}^*\Big)^* \subseteq \Big(\mathcal{R}^* \circ \big(\mathcal{F}^*\big)^{-1}\Big)^*$.

Proof. The check the equivalences of the assertions (2)–(5) of this theorem to assertions (2)–(5) of Theorem 7.3, it is convenient to use that $*$ is an inversion and composition compatible closure operation for relators. Thus,

(a) $\big(\mathcal{R}^*\big)^{-1} = \big(\mathcal{R}^{-1}\big)^*$ for any relator $\mathcal{R}$ on X to Y;

(b) $\mathcal{R} \subseteq S^* \iff \mathcal{R}^* \subseteq \mathcal{S}^*$ for any relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y;

(c) $\big(\mathcal{S} \circ \mathcal{R}\big)^* = \big(\mathcal{S}^* \circ \mathcal{R}^*\big)^*$ for any relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Remark 7.10. Note that in Theorems 7.3, 7.4 and 7.9, R and S may be thought of not only as certain order relations $\leq_X$ and $\leq_Y$, but also as some surroundings $B_\delta^{d_X}$ and $B_\varepsilon^{d_Y}$.

Therefore, instead of the term "increasing", we can equally well use the term "continous". Namely, if $R = B_\delta^{d_X}$ and $S = B_\varepsilon^{d_Y}$, then assertion (2) of Theorem 7.3 means only that $d_X(u, v) < \delta$ implies $d_Y\big(f(u), f(v)\big) < \varepsilon$.

8. Four basic continuity properties for pairs of relators

Now, by pexiderizing the inclusions (2)–(5) in Theorem 7.9, we may naturally introduce the following general definition whose origins go back to [48, 39, 61, 91].

Definition 8.1. Assume that

(a) $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces;

(b) $\mathcal{F}$ is a relator on X to Z and $\mathcal{G}$ be a relator on Y to W;

(c) $\square = (\square_i)_{i=1}^{6}$ is a family of direct unary operations for relators.

Then, we say that the pair $(\mathcal{F}, \mathcal{G})$ is, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$,

(1) *upper* $\square$*–semicontinuous* if

$$\left(\mathcal{S}^{\square_1} \circ \mathcal{F}^{\square_2}\right)^{\square_3} \subseteq \left(\mathcal{G}^{\square_4} \circ \mathcal{R}^{\square_5}\right)^{\square_6},$$

(2) *mildly* $\square$*–continuous* if

$$\left(\left(\mathcal{G}^{\square_1}\right)^{-1} \circ \mathcal{S}^{\square_2} \circ \mathcal{F}^{\square_3}\right)^{\square_4} \subseteq \mathcal{R}^{\square_5 \square_6},$$

(3) *vaguely* $\square$*–continuous* if

$$\mathcal{S}^{\square_1 \square_2} \subseteq \left(\mathcal{G}^{\square_3} \circ \mathcal{R}^{\square_4} \circ \left(\mathcal{F}^{\square_5}\right)^{-1}\right)^{\square_6},$$

(4) *lower* $\square$*–semicontinuous* if

$$\left(\left(\mathcal{G}^{\square_1}\right)^{-1} \circ \mathcal{S}^{\square_2}\right)^{\square_3} \subseteq \left(\mathcal{R}^{\square_4} \circ \left(\mathcal{F}^{\square_5}\right)^{-1}\right)^{\square_6}.$$

Remark 8.2. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, one can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & Z \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle S} \\ Y & \xrightarrow{G} & W \end{array}$$

Remark 8.3. Now, for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) may, for instance, be naturally called upper $\square$–semicontinuous, if the pair $(\{F\}, \{G\})$ is upper $\square$–semicontinuous. That is,

$$\left(\mathcal{S}^{\square_1} \circ \{F\}^{\square_2}\right)^{\square_3} \subseteq \left(\{G\}^{\square_4} \circ \mathcal{R}^{\square_5}\right)^{\square_6}.$$

Unfortunately, this condition may greatly differ from the more natural requirement that $\left(\mathcal{S}^{\square_1} \circ F\right)^{\square_3} \subseteq \left(G \circ \mathcal{R}^{\square_5}\right)^{\square_6}$ which should also be given an appropriate name.

In this respect, it is worth noticing that, for instance, we have

$$\{F\}^{\#} = \{F\}^{\wedge} = \{F\}^{*} \qquad \text{and} \qquad \{F\}^{\vartriangle} = \left\{F \circ X^X\right\}^{*}$$

for all $F \in \mathcal{F}$.

Remark 8.4. Thus, the the pair (F, G) may, for instance, be naturally called selectionally upper $\square$–semicontinuous if for any selection functions f of F and g of G the pair (f, g) is upper $\square$–semicontinuous.

Moreover, the pair $(\mathcal{F},\ \mathcal{G})$ may, for instance, be naturally called *elementwise upper* $\square$*–semicontinuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair $(F,\ G)$ is upper $\square$–semicontinuous. This may greatly differ from property (1).

Remark 8.5. If in particular $\square$ is a direct unary operation for relators, then the pair $(\mathcal{F},\ \mathcal{G})$ may, for instance, be also naturally called upper $\square$–semicontinuous if it is upper $\big(\square\big)_{i=1}^{6}$–semicontinuous. That is,

$$\left(\mathcal{S}^{\square} \circ \mathcal{F}^{\square}\right)^{\square} \subseteq \left(\mathcal{G}^{\square} \circ \mathcal{R}^{\square}\right)^{\square} .$$

Remark 8.6. Thus, the pair $(\mathcal{F},\ \mathcal{G})$ may, for instance, be naturally called *properly upper semicontinuous* if it is upper $\square$–semicontinuous with $\square$ being the identity operation for relators. That is, $\mathcal{S} \circ \mathcal{F} \subseteq \mathcal{G} \circ \mathcal{R}$.

Moreover, the pair $(\mathcal{F},\ \mathcal{G})$ may, for instance, be also naturally called *uniformly, proximally, topologically and paratopologically upper semicontinuous* if it is $\square$–semicontinuous with $\square = *,\ \#,\ \wedge$ and $\vartriangle$, respectively.

Thus, by using the operations $\square\infty$ and $\square\partial$ instead of $\square$, we can quite similarly speak of the corresponding quasi upper semicountinuity and pseudo upper semicontinuity properties of $(\mathcal{F},\ \mathcal{G})$.

Remark 8.7. Finally, we note that if in particular $X = Y$ and $Z = W$, then the relator $\mathcal{F}$ and a relation $F \in \mathcal{F}$ may, for instance, be naturally called upper $\square$–semicontinuous if the pairs $(\mathcal{F},\ \mathcal{F})$ and $(F,\ F)$ are upper $\square$–semicontinuous, respectively.

9. Galois and Pataki connections

By our former papers [69, 77], we may naturally use the following

Definition 9.1. Assume that

(a) $X(R)$ and $Y(S)$ are gosets;

(b) φ is a function of X to itself;

(c) f is a function of X to Y and g is a function of Y to X.

Then, we say that f is *increasingly right*

(1) *g–seminormal* if for all $x \in X$ and $y \in Y$

$$f(x)\, S\, y \quad \Longrightarrow \quad x\, R\, g(y).$$

(2) *φ–semiregular* if for all $u,\, v \in X$

$$f(u)\, S\, f(v) \quad \Longrightarrow \quad u\, R\, \varphi(v).$$

Remark 9.2. If property (1) holds, then we may also say that f and g form an *increasing right Galois semiconnection* between $X(R)$ and $Y(S)$.

On the other hand, if property (2) holds, then we may also say that f and φ form an increasing right Pataki semiconnection between $X(R)$ and $Y(S)$.

The corresponding increasing left seminormality and semiregularity properties of f can be defined by reversing the implications in properties (1) and (2).

Moreover, the function f may, for instance, be naturally called *increasingly g–normal* if it is increasingly left and right g–seminormal.

By using the above definitions, we can easily prove the following three theorems.

Theorem 9.3. *If f is an increasingly left g–seminormal function of $X(R)$ to $Y(S)$, then g is an increasingly right f–seminormal function of $Y(S^{-1})$ to $X(R^{-1})$.*

Proof. For any $y \in Y$ and $x \in X$

$$g(y)\, R^{-1}\, x \implies x\, R\, g(y) \implies f(x)\, S\, y \implies y\, S^{-1}\, f(x)\,.$$

Theorem 9.4. *If f is increasingly right g–seminormal and $\varphi = g \circ f$, then f is increasingly right φ–semiregular.*

Theorem 9.5. *If f is increasingly right φ–semiregular, f is onto Y, and $\varphi = g \circ f$, then f is increasingly right g–seminormal.*

Proof. Suppose that $x \in X$ and $y \in Y$. Then, since $Y = f[X]$, there exists $v \in X$ such that $y = f(v)$. Thus, we can easily see that

$$\begin{aligned} f(x)\, S\, y &\implies f(x)\, S\, f(v) \implies x\, R\, \varphi(v) \\ &\implies x\, R\, (g \circ f)(v) \implies x\, R\, g\big(f(v)\big) \implies x\, R\, g(y)\,. \end{aligned}$$

Remark 9.6. By Theorem 9.4, it is clear that several properties of the increasingly normal functions can be immediately derived from those of the increasingly regular ones. Therefore, the latter ones have to studied before the former ones.

While, from Theorem 9.5, we can see that the increasing regular functions are still less general objects than the increasingly normal ones. Actually, they are strictly between closure operations and increasingly normal functions.

Namely, concerning them we can also prove the following

Theorem 9.7. *If R is a preorder, then the following assertions are equivalent:*

(1) *φ is a closure operation;* (2) *φ is increasingly φ–regular;*

(3) *there exists an increasingly φ–regular function h of $X(R)$ to a proset $Z(T)$.*

Hence, by using the *induced order (preference) relation* Ord_f, defined such that

$$\operatorname{Ord}_f(u) = \big\{\, v \in X : \quad f(u)\, S\, f(v) \,\big\}$$

for all $u \in X$, we can easily derive the following

Corollary 9.8. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly φ–regular;*

(2) *φ is a closure operation and $\operatorname{Ord}_\varphi = \operatorname{Ord}_f$.*

Finally, we note that, concerning normal functions, the following three theorems are also true.

Theorem 9.9. *If R is a preorder, then the following assertions are equivalent:*

(1) *φ is an involution operation;* (2) *φ is φ–normal.*

Theorem 9.10. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is g–normal;*

(2) *f and g are increasing, $g \circ f$ is extensive and $f \circ g$ is intensive.*

Remark 9.11. This theorem shows that the recent definition of Galois connections [11, p. 155], suggested by Schmidt [45, p. 209], is equivalent to the old one given by Ore [33].

Theorem 9.12. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is g–normal;*

(2) *f is increasing and $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$.*

Remark 9.13. Here, the induced interior relation Int_f is defined such that

$$\operatorname{Int}_f(y) = \{ x \in X : \quad f(x)\, S\, y \}$$

for all $y \in Y$. Thus, we have $\operatorname{Ord}_f = \left(\operatorname{Int}_f \circ f \right)^{-1}$.

Moreover, we can also easily show that f is increasingly g–normal if and only if $\operatorname{Int}_f(y) = \operatorname{lb}\big(g(y)\big)$ for all $y \in Y$, even if R and S are not suppose to have any particular property.

10. Relational characterizations of increasingly right seminormal functions

By using Definition 9.1, analogously to Theorem 7.4, we can also prove

Theorem 10.1. *The following assertions are equivalent:*

(1) f *is increasingly right* g*–seminormal;*

(2) $S \circ f \subseteq g^{-1} \circ R\,;$ (3) $g \circ S \circ f \subseteq R\,.$

Proof. For any $x \in X$ and $y \in Y$, the following assertions are equivalent:

$$f(x)\, S\, y \implies x\, R\, g(y)\,,$$
$$y \in S\big(f(x)\big) \implies g(y) \in R(x)\,,$$
$$y \in S\big(f(x)\big) \implies y \in g^{-1}[\,R(x)\,]\,,$$
$$S\big(f(x)\big) \subseteq g^{-1}[\,R(x)\,]$$
$$\big(S \circ f\big)(x) \subseteq \big(g^{-1} \circ R\big)(x)\,.$$

Hence, by Definition 9.1, we can see that assertions (1) and (2) are equivalent.

Moreover, by using some basic properties of composition, we can see that

$$(2) \implies g \circ S \circ f \subseteq g \circ g^{-1} \circ R \implies g \circ S \circ f \subseteq \Delta_X \circ R \implies (3)$$

and

$$(3) \implies g^{-1} \circ g \circ S \circ f \subseteq g^{-1} \circ R \implies \Delta_Y \circ S \circ f \subseteq g^{-1} \circ R \implies (2)\,.$$

Therefore, assertions (2) and (3) are also equivalent.

From this theorem, by using some basic theorems on the box product [74], we can derive the following

Corollary 10.2. *The following assertions are equivalent:*

(1) f *is increasingly right* g*–seminormal;*

(2) $\big(f^{-1} \boxtimes S\big)[\,\Delta_Y\,] \subseteq \big(R^{-1} \boxtimes g^{-1}\big)[\,\Delta_X\,]\,;$ (3) $\big(f^{-1} \boxtimes g\big)[\,S\,] \subseteq R\,.$

Remark 10.3. From Theorem 10.1, by using the operation $*$, we can easily derive an analogue of Theorem 7.9.

However, it is now more important to note that, by using Theorem 10.1 and the operation $\circledast = c * c$, we can also prove the following

Theorem 10.4. *Under the notations*

$$\mathcal{F} = \{f\}\,, \qquad \mathcal{G} = \{g\}\,, \qquad \mathcal{R} = \{R\}\,, \qquad \mathcal{S} = \{S\}\,,$$

the following assertions are equivalent:

(1) f *is increasingly right* g*–normal;*

(2) $\big(\mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast}\big)^{\circledast} \subseteq \Big(\big(\mathcal{G}^{\circledast}\big)^{-1} \circ \mathcal{R}^{\circledast}\Big)^{\circledast}\,;$ (3) $\big(\mathcal{G}^{\circledast} \circ \mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast}\big)^{\circledast} \subseteq \mathcal{R}^{\circledast\circledast}\,.$

Remark 10.5. To check this, note that for any relator $\mathcal{R}$ and relation S on X to Y we have

$$S \in \mathcal{R}^{\circledast} \iff S \in \mathcal{R}^{c*c} \iff S^c \in \mathcal{R}^{c*} \iff \exists\, U \in \mathcal{R}^c : \ U \subseteq S^c$$
$$\iff \exists\, R \in \mathcal{R} : \ R^c \subseteq S^c \iff \exists\, R \in \mathcal{R} : \ S \subseteq R .$$

Therefore,

$$\mathcal{R}^{\circledast} = \big\{ S \subseteq X \times Y : \ \exists\ R \in \mathcal{R} : \ S \subseteq R \big\} = \textstyle\bigcup_{R \in \mathcal{R}} \mathcal{P}(R) .$$

11. Relational characterizations of increasingly right semiregular functions

By using Definition 9.1, analogously to Theorem 10.1, we can also prove

Theorem 11.1. *The following assertions are equivalent:*

(1) f *is increasingly right* φ*–semiregular;*

(2) $f^{-1} \circ S \circ f \subseteq \varphi^{-1} \circ R$; (3) $\varphi \circ f^{-1} \circ S \circ f \subseteq R$.

Proof. For any $u, v \in X$, the following assertions are equivalent:

$$f(u) \, S \, f(v) \implies u \, R \, \varphi(v) ,$$
$$f(v) \in S\big(f(u)\big) \implies \varphi(v) \in R(u) ,$$
$$v \in f^{-1}\big[S\big(f(u)\big) \big] \implies v \in \varphi^{-1}\big[R(u) \big] ,$$
$$f^{-1}\big[S\big(f(u)\big) \big] \subseteq \varphi^{-1}\big[R(u) \big]$$
$$\big(f^{-1} \circ S \circ f \big)(u) \subseteq \big(\varphi^{-1} \circ R \big)(u) .$$

Therefore, by Definition 9.1, assertions (1) and (2) are equivalent.

Moreover, by using some basic properties of composition, we can see that

$$(2) \implies \varphi \circ f^{-1} \circ S \circ f \subseteq \varphi \circ \varphi^{-1} \circ R \implies \varphi \circ f^{-1} \circ S \circ f \subseteq \Delta_X \circ R \implies (3)$$

and

$$(3) \implies \varphi^{-1} \circ \varphi \circ f^{-1} \circ S \circ f \subseteq \varphi^{-1} \circ R \implies \Delta_X \circ f^{-1} \circ S \circ f \subseteq \varphi^{-1} \circ R \implies (2).$$

Therefore, assertions (2) and (3) are also equivalent.

From this theorem, by using some basic theorems on the box product [74], we can derive the following

Corollary 11.2. *The following assertions are equivalent:*

(1) f *is increasingly right* φ*–semiregular;*

(2) $\big(f^{-1} \boxtimes f^{-1} \big)[S] \subseteq \big(R^{-1} \boxtimes \varphi^{-1} \big)[\Delta_X]$; (3) $\varphi \circ \big(f^{-1} \boxtimes f^{-1} \big)[S] \subseteq R$.

Remark 11.3. From Theorem 11.1, by using the operation $*$, we can easily derive an analogue of Theorem 7.9

However, again it is more important to note that, by using Theorem 11.1 and the operation $\circledast = c * c$, we can also prove the following

Theorem 11.4. *Under the notations*

$$\mathcal{F} = \{ f \} , \qquad \Phi = \{ \varphi \} , \qquad \mathcal{R} = \{ R \} \qquad \mathcal{S} = \{ S \} ,$$

the following assertions are equivalent:

(1) f *is increasingly right* φ*–regular;*

(2) $\Big(\big(\mathcal{F}^{\circledast} \big)^{-1} \circ \mathcal{S}^{\circledast} \circ \mathcal{F}^{\circledast} \Big)^{\circledast} \subseteq \Big(\big(\Phi^{\circledast} \big)^{-1} \circ \mathcal{R}^{\circledast} \Big)^{\circledast}$.

12. Some increasing right seminormality and semiregularity properties for pairs of relators

Now, analogously to Definition 8.1, we may also naturally introduce the following

Definition 12.1. Assume that

(a) $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces;

(b) $\mathcal{F}$ is a relator on X to Z and $\mathcal{G}$ be a relator on Y to W;

(c) $\Box = \big(\Box_i\big)_{i=1}^{6}$ is a family of direct unary operations for relators.

Then, we say that the relator $\mathcal{F}$ is, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$,

(1) *increasingly upper right* $\Box$*–*$\mathcal{G}$*–seminormal* if

$$\Big(\mathcal{S}^{\Box_1} \circ \mathcal{F}^{\Box_2}\Big)^{\Box_3} \subseteq \bigg(\Big(\mathcal{G}^{\Box_4}\Big)^{-1} \circ \mathcal{R}^{\Box_5}\bigg)^{\Box_6},$$

(2) *increasingly mildly right* $\Box$*–*$\mathcal{G}$*–seminormal* if

$$\Big(\mathcal{G}^{\Box_1} \circ \mathcal{S}^{\Box_2} \circ \mathcal{F}^{\Box_3}\Big)^{\Box_4} \subseteq \mathcal{R}^{\Box_5 \Box_6},$$

(3) *increasingly vaguely right* $\Box$*–*$\mathcal{G}$*–seminormal* if

$$\mathcal{S}^{\Box_1 \Box_2} \subseteq \bigg(\Big(\mathcal{G}^{\Box_3}\Big)^{-1} \circ \mathcal{R}^{\Box_4} \circ \Big(\mathcal{F}^{\Box_5}\Big)^{-1}\bigg)^{\Box_6},$$

(4) *increasingly lower right* $\Box$*–*$\mathcal{G}$*–seminormal* if

$$\Big(\mathcal{G}^{\Box_1} \circ \mathcal{S}^{\Box_2}\Big)^{\Box_3} \subseteq \bigg(\mathcal{R}^{\Box_4} \circ \Big(\mathcal{F}^{\Box_5}\Big)^{-1}\bigg)^{\Box_6}.$$

Remark 12.2. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, one can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xleftarrow{\;G\;} & W \end{array}$$

Thus, for instance, we can easily establish the following

Theorem 12.3. *If in particular the operation* $\Box_4$ *is inversion compatible, then the following assertions are equivalent:*

(1) $\big(\mathcal{F}, \mathcal{G}^{-1}\big)$ *is upper* $\Box$*–continuous;*

(2) $\mathcal{F}$ *is increasingly upper right* $\Box$*–*$\mathcal{G}$*–normal.*

Now, in contrast to Definition 12.1, we can only introduce the following

Definition 12.4. Assume that

(a) $(X, Y)(\mathcal{R})$ and $Z(\mathcal{S})$ are relator spaces;

(b) $\mathcal{F}$ is a relator on X to Z and $\mathbf{\Phi}$ is a relator on X to Y;

(c) $\Box = (\Box_i)_{i=1}^{7}$ is a family of direct unary operations for relators.

Then, we say that the relator $\mathcal{F}$ is *increasingly right* $\Box$*–*$\mathbf{\Phi}$*–semiregular*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\bigg(\Big(\mathcal{F}^{\Box_1}\Big)^{-1} \circ \mathcal{S}^{\Box_2} \circ \mathcal{F}^{\Box_3}\bigg)^{\Box_4} \subseteq \bigg(\Big(\Phi^{\Box_5}\Big)^{-1} \circ \mathcal{R}^{\Box_6}\bigg)^{\Box_7}.$$

Remark 12.5. To keep in mind the above assumptions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $\Phi \in \mathbf{\Phi}$, one can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & Z \\ {\scriptstyle \Phi}\big\downarrow {\scriptstyle R} & & \big\downarrow {\scriptstyle S} \\ Y & & Z \end{array}$$

Thus, for instance, we can easily establish the following

Theorem 12.6. *If in particular $\Box_5 = \Box_6 = \Box_7$ is an inversion and composition compatible closure operation for relators and $\Diamond = \big(\Box_i\big)_{i=1}^{6}$, then the following assertions are equivalent:*

(1) *$\mathcal{F}$ is mildly $\Diamond$–continuous with respect to the relators $\Phi \circ \mathcal{R}$ and $\mathcal{S}$;*

(2) *$\mathcal{F}$ is increasingly right $\Diamond$–$\mathbf{\Phi}$–semiregular with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.*

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